Pre-modern (or *cossic*) algebra

Units were counted in *dirhams*, a unit of currency, or sometimes as *āhād* (“units”) or *min al-ʿadad* (“in number”), and often the name was dropped altogether. So “ten dirhams”, “ten units”, “ten in number” and “ten” all took the meaning of the number ten. (Units were the one exception to the how-to-interpretation-equations rule we just stated above; though al-Khwārizmī usually gave units in *dirhams*.)

The terms *jidhr* and *māl* were borrowed from arithmetical problem solving, where the *māl* is literally an amount of money or an abstract quantity, and the “root” is its square root. Many problems in Arabic arithmetic asked for an unknown *māl* satisfying some condition.

The following example of an equation using algebraic notation was given by the Moroccan mathematician Ibn Ghazi al-Miknasi (1437–1513CE) in his 1471 text *Aim of the Students in Commentary on Desire of Reckoners*:

> Five *māls* and four things and three in number equal two *māls* and three things and six in number, and its figure is
>
> $$\begin{array}{ccc} m & t & \\ 5 & 4 & 3 \end{array} = \begin{array}{ccc} m & t & \\ 2 & 3 & 6 \end{array}$$

The symbolic expression shown here is a transcription into modern Western characters and left-right order, of Ibn Ghazi’s original expression, which was written right-to-left, using Arabic symbols. The m represents *māls*, the t stands for “things”. Notice how this notation keeps the different kinds of terms separate, in columns, much like a modern spreadsheet.

Starting from the left in the expression as shown above, in his presentation Ibn Ghazi represented the “five *māls*” as the letter *mīm* (the first letter in *māl*) above a 5 (here shown as an m above a 5). Moving to the second term, he represented the “four things” of the problem as a *shīn* (the first letter in *shayʾ*, “thing”) above a 4 (here shown as a t above a 4). And the problem’s “three in number” is simply represented as a “3”. Similarly for the other side to the equation.

Where we show an equals sign, Ibn Ghazi used an elongated letter *lām*, the last (written) letter in the Arabic word used to indicate equality. (It actually looks like a backwards letter “L”.) The polynomial that appears to the right of the equality sign in our figure follows the same scheme.

Here is Ibn Ghazi’s equation in modern algebraic notation:

$$5x^2 + 4x + 3 = 2x^2 + 3x + 6$$

This looks like a syntactic rearrangement/transcription of the various terms in Ibn Ghazi’s equation, but in fact there was much more going on here.

To readers today, the modern notation we see above carries with it a concept of polynomials that differs from medieval times. Mathematicians back then did not conceive of polynomials like "five *māls* and four things and three in number" the way we do today, as linear combinations, with the implied operations of addition, subtraction, and scalar multiplication. They were instead seen as aggregations of the species, or names, of the powers of the unknown, with no operations involved.

This follows from their concept of number as an amount or measure *of something*. All numbers had a species. In a term like "five *māls*" the "five" is not a scalar multiple, but tells us instead how many *māls* there are. It is like saying "five bananas". (No multiplication there.) The "five" was called the "number" (*ʿadad*) of the term. There was no special word for "coefficient".

The polynomial "five *māls* and four *things* and three in *number*" is a collection or aggregation of twelve items of three species. Like saying "five bananas and four apples and three pears". The "and" (*wa* in the Arabic equation) between the terms in the text is the common conjunction, and does not have the meaning of the modern "plus".

Notice too that there is no symbol like our "+" in Ibn Ghazi's equation. The terms are merely placed next to each other. (Diophantus did not use any word for "and" either, and it was absent in some medieval Italian books.)

In fact, the word *wa* ("and") was not always used in displayed notation, though it was always used in the running text. When it was, it was usually to distinguish it from the notation for "less" (*illā*) in a problem that in modern notation would yield a polynomial with both positive and negative coefficients. Again, an Arabic expression such as "ten things less a *māl*" did not express the arithmetical operation of subtraction; rather, the expression was taken to denote a "deficient ten things".

Viewing and handling the numbers (often *dirhams*) and the different powers of the unknown x, (the *shayʾ* = "thing", or sometimes *jidhr* = "root") and its square x^2 (*māl* = sum of money or possession), meant that the logical relationships between the terms was inaccessible to them. Yet those relationships lie at the core of modern algebraic calculation. This meant that, although from today's perspective we can view their work as (an early form of) "algebra", in reality what they were doing was *arithmetic* with an unknown (the purpose being to determine the value of that unknown). The Italian word for an arithmetic unknown was *cosa* (thing), and as a result the algebra that was practiced by the medieval Arabic mathematicians is sometimes referred to as "*cossic* algebra".